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Notes on Plane Partitions, V

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1. INTRODUCTION

In the classic theory of partitions, there are three arithmetic functions which play an especially important role:

$p(n)$ (the total number of partitions of n),
$q(n)$ (the number of partitions of n into distinct parts), and
$q_0(n)$ (the number of partitions of n into distinct odd parts).

Their generating functions,

$$P(x) = \sum_{n=0}^{\infty} p(n)\, x^n, \qquad Q(x) = \sum_{n=0}^{\infty} q(n)\, x^n$$

and

$$Q_0(x) = \sum_{n=0}^{\infty} q_0(n)\, x^n,$$

are easily seen to be given by

$$P(x) = \prod_{\nu=1}^{\infty} (1 - x^{\nu})^{-1}, \tag{1}$$

$$Q(x) = \prod_{\nu=1}^{\infty} (1 - x^{2\nu-1})^{-1} = \prod_{\nu=1}^{\infty} (1 + x^{\nu}), \tag{2}$$

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and

$$Q_0(x) = \prod_{\nu=1}^{\infty} (1 + x^{2\nu-1}); \tag{3}$$

here, and in what follows, the identities are valid for $|x| < 1$. MacMahon [2] found the analog of (1) for the plane partition function $a(n)$. He proved that its generating function $A(x) = \sum_{n=0}^{\infty} a(n)\, x^n$ is given by

$$A(x) = \prod_{\nu=1}^{\infty} (1 - x^{\nu})^{-\nu}. \tag{4}$$

In [1] a suitable planar analog of (2) was found; it was shown there that, if $b(n)$ is the number of plane partitions of n whose parts decrease strictly along each row, then

$$\begin{aligned} B(x) &= \sum_{n=0}^{\infty} b(n)\, x^n \\ &= \prod_{\nu=1}^{\infty} (1 - x^{\nu})^{-[(\nu+1)/2]}. \end{aligned} \tag{5}$$

In the present note we obtain a planar analog of (3). More specifically, it will be shown that, if $b_0(n)$ is the number of plane partitions of n into odd parts which decrease strictly along each row, then

$$\begin{aligned} B_0(x) &= \sum_{n=0}^{\infty} b_0(n)\, x^n \\ &= \prod_{\nu=1}^{\infty} (1 - x^{2\nu-1})^{-1} (1 - x^{2\nu})^{-[\nu/2]} = Q_0(x)\, B(x^2). \end{aligned} \tag{6}$$

We can also interpret $b_0(n)$ as the number of symmetric plane partitions of n, i.e., plane partitions of the form $n = \sum n_{i,j}$ with $n_{i,j} = n_{j,i}$. The proof of this remark is a straightforward generalization of the familiar argument which shows that $q_0(n)$ is the number of self-conjugate partions of n.

As is usual in the study of plane partitions, we approach the proof of (6) by considering the number $b_{0k}(n)$ of k-rowed partitions of n into odd parts which decrease strictly along each row. For the generating function $B_{0k} = \sum_{n=0}^{\infty} b_{0k}(n)\, x^n$ we obtain the expressions:

$$B_{0k}(x) = P(x^2)^{k/2} \prod_{\nu=1}^{k/2} (1 + x^{2\nu-1}) \prod_{\nu=1}^{k-2} (1 - x^{2\nu})^{[(k-\nu)/2]} \tag{7}$$

if k is even,

$$B_{0k}(x) = Q_0(x)\, P(x^2)^{(k-1)/2} \frac{\prod_{\nu=1}^{k-1} (1 - x^{2\nu})^{[(k+1-\nu)/2]}}{\prod_{\nu=1}^{(k-1)/2} (1 - x^{2\nu})} \tag{8}$$

if k is odd.

It is easily seen that both of these equations approach (6) as $k \to \infty$.

2. Two Lemmas on Determinants

The derivation of equations (7) and (8) involves calculating certain determinants. To carry out these computations we require the results of this section.

Lemma 1. *Let $S = (s_{i,j})$ be a skew-symmetric Toeplitz matrix of even order $2m$, so that $s_{i,j} = e_{j-i}$, where $e_{-\nu} = -e_\nu$ ($\nu = 0, 1,..., 2m - 1$). Then the Pfaffian of S is equal to the determinant of the $m \times m$ matrix $T = (t_{i,j})$, where*

$$t_{i,j} = e_{|i-j|+1} + e_{|i-j|+3} + \cdots + e_{i+j-1}\,.$$

For example, if $m = 3$, Lemma 1 asserts that the Pfaffian of

$$\begin{bmatrix} 0 & e_1 & e_2 & e_3 & e_4 & e_5 \\ -e_1 & 0 & e_1 & e_2 & e_3 & e_4 \\ -e_2 & -e_1 & 0 & e_1 & e_2 & e_3 \\ -e_3 & -e_2 & -e_1 & 0 & e_1 & e_2 \\ -e_4 & -e_3 & -e_2 & -e_1 & 0 & e_1 \\ -e_5 & -e_4 & -e_3 & -e_2 & -e_1 & 0 \end{bmatrix}$$

is equal to

$$\begin{vmatrix} e_1 & e_2 & e_3 \\ e_2 & e_1 + e_3 & e_2 + e_4 \\ e_3 & e_2 + e_4 & e_1 + e_3 + e_5 \end{vmatrix}.$$

Proof of Lemma 1. To the last column of S add the 2nd, 4th, 6th,..., and $(2m - 2)$nd columns. Then to the $(2m - 1)$st column of S add the 3rd, 5th, 7th,..., and $(2m - 3)$rd columns. In general, if $0 \leqslant j \leqslant m - 2$, to the $(2m - j)$th column of S we add columns $j + 2, j + 4$,..., and $2m - j - 2$. Call the resulting matrix S^*. Now perform the same operations on the rows of S^*, i.e., to the $(2m - j)$th row of S^* (where $0 \leqslant j \leqslant m - 2$), add the $(j + 2)$nd, $(j + 4)$th,..., and $(2m - j - 2)$nd rows. The resulting matrix S^{**} is easily seen to be

$$S^{**} = \begin{vmatrix} \hat{S} & T \\ -T & 0 \end{vmatrix},$$

where $\hat{S}$ is the $m \times m$ matrix $(s_{i,j})$, $1 \leqslant i, j \leqslant m$. It follows from this that $\det S = \det S^{**} = (\det T)^2$. Therefore the Pfaffian of S is $\pm \det T$. Since both the Pfaffian of S and $\det T$ contain a term $e_1{}^m$, the plus sign holds, and the proof is complete.

LEMMA 2. *If $x_1, \ldots, x_m$ are indeterminates, then the $m \times m$ determinant*

$$D = \begin{vmatrix} x_1^{m-1} & x_1^{m-2} + x_1{}^m & x_1^{m-3} + x_1^{m+1} \cdots 1 + x_1^{2m-2} \\ x_2^{m-1} & x_2^{m-2} + x_2{}^m & x_2^{m-3} + x_2^{m+1} \cdots 1 + x_2^{2m-2} \\ \vdots & & \\ x_m^{m-1} & x_m^{m-2} + x_m{}^m & x_m^{m-3} + x_m^{m+1} \cdots 1 + x_m^{2m-2} \end{vmatrix}$$

is equal to

$$\prod_{i<j} (x_i - x_j)(1 - x_i x_j).$$

Proof. The determinant D is clearly a polynomial in $x_1, \ldots, x_m$ of degree

$$(m-1) + m + (m+1) + \cdots + (2m-2) = \frac{3m(m-1)}{2}.$$

Choose two integers i, j with $1 \leqslant i < j \leqslant m$. If $x_i = x_j$, two rows of D become equal, so D vanishes. Hence D is divisible by $x_i - x_j$. If the j-th row of D is divided by x_j^{2m-2} it becomes

$$(x_j^{-m-1}, x_j^{-m} + x_j^{-m+2}, x_j^{-m-1} + x_j^{-m+3}, \ldots, x_j^{-2m+2} + 1),$$

which is the same as the original j-th row with x_j replaced by x_j^{-1}. Hence D vanishes whenever $x_i = x_j^{-1}$, and therefore D is divisible by $1 - x_i x_j$. From these considerations we see that D is divisible by

$$P = \prod_{i<j} (x_i - x_j)(1 - x_i x_j).$$

Now P is a polynomial of degree $3m(m-1)/2$ and therefore the quotient D/P is a constant. Since both D and P contain the term

$$x_1^{m-1} x_2^{m-2} x_3^{m-3} \cdots x_{m-1},$$

the constant D/P is unity. Thus $D = P$, as was to be shown.

COROLLARY. *The determinant of the matrix*

$$E(x_1, \ldots, x_n) = \begin{bmatrix} 1 & x_1^{-1} + x_1 & x_1^{-2} + x_1^{2} & \cdots & x_1^{-m+1} + x^{m-1} \\ x_2 & 1 + x_2^{2} & x_2^{-1} + x_2^{3} & \cdots & x_2^{-m+2} + x_2^{m} \\ \vdots & & & & \\ x_m^{m-1} & x_m^{m-2} + x_m^{m} & x_m^{m-1} + x_m^{m+1} & \cdots & 1 + x_m^{2m-2} \end{bmatrix}$$

is equal to $\prod_{i<j} (1 - x_i^{-1}x_j)(1 - x_i x_j)$.

Proof. det E is obtained from D by multiplying the ν-th row by $x_\nu^{-m+\nu}$. Hence

$$\begin{aligned} \det E &= D \prod_{\nu=1}^{m-1} x_\nu^{-m+\nu} \\ &= \prod_{i<j} (x_i - x_j)(1 - x_i x_j) \prod_{\nu=1}^{m-1} x_\nu^{-m+\nu} \\ &= \prod_{i<j} (1 - x_i^{-1} x_j)(1 - x_i x_j). \end{aligned}$$

3. EXPRESSION OF $B_{0k}(x)$ AS A DETERMINANT

If $f_1, \ldots, f_k$ are non-negative integers satisfying $f_1 \geqslant f_2 \geqslant \cdots \geqslant f_k$, then we denote by $b_0(n; f_1, \ldots, f_k)$ the number of k-rowed partitions of n with exactly f_i non-zero parts on the i-th row ($i = 1, \ldots, k$), where the parts are odd integers decreasing strictly along each row. Put

$$B_0(x; f_1, \ldots, f_k) = \sum_{n=0}^{\infty} b_0(n; f_1, \ldots, f_k)\, x^n.$$

Clearly

$$B_{0k}(x) = \sum_{f_1 \geqslant \cdots \geqslant f_k \geqslant 0} B_0(x; f_1, \ldots, f_k).$$

We will first express these quantities in terms of the analogous functions $b(n; f_1, \ldots, f_k)$ and $B(x; f_1, \ldots, f_k)$, where the parts are not required to be odd. To do this, let $f = f_1 + \cdots + f_k$. If $n = \sum n_{i,j}$ is any partition of n of the type enumerated by $b(n; f_1, \ldots, f_k)$, then $2n - f = \sum (2n_{i,j} - 1)$ is a partition of $2n - f$ of the type enumerated by $b_0(2n - f; f_1, \ldots, f_k)$. Since this map is clearly invertible, we have $b(n; f_1, \ldots, f_k) = b_0(2n - f; f_1, \ldots, f_k)$. Multiplying both sides by x^{2n-f} and summing over all $n \geqslant 0$, we obtain

$$B(x^2; f_1, \ldots, f_k)\, x^{-f} = B_0(x; f_1, \ldots, f_k).$$

In [1] it was shown that, if $h_j = f_j + k - j$ $(j = 1,..., k)$, then $B(x; f_1,..., f_k) = \det(\eta_{i-k+h_j})$, where

$$\eta_\nu = \frac{x^{\binom{\nu+1}{2}}}{(1-x)(1-x^2)\cdots(1-x^\nu)}$$

for $\nu > 0$, $\eta_0 = 1$, and $\eta_\nu = 0$ for $\nu < 0$.

Put

$$\zeta_\nu = \frac{x^{\nu^2}}{(1-x^2)(1-x^4)\cdots(1-x^{2\nu})},$$

so that $\zeta_\nu(x) = \eta_\nu(x^2)\, x^{-\nu}$. Then

$$\det(\zeta_{i-k+h_j}(x)) = \det(\eta_{i-k+h_j}(x^2)\, x^{-i+k-h_j}).$$

From the i-th row of this determinant we can remove the factor x^{-i}, and from the j-th column we can remove the factor $x^{k-h_j} = x^{j-f_j}$. The product of all the factors thus removed is $x^{-f_1-\cdots-f_k} = x^{-f}$, from which we see that

$$\begin{aligned}\det(\zeta_{i-k+h_j}) &= x^{-f}\det(\eta_{i-k+h_j}(x^2))\\ &= x^{-f}B(x^2; f_1,..., f_k)\\ &= B_0(x; f_1,..., f_k).\end{aligned}$$

Therefore

$$\begin{aligned}B_0(x) &= \sum_{f_1\geqslant\cdots\geqslant f_k} B_0(x; f_1,..., f_k)\\ &= \sum_{h_1>\cdots>h_k} \det(\zeta_{i-k+h}),\end{aligned}$$

where the summation is now over all integers $h_1 > h_2 > \cdots > h_k \geqslant 0$. By virtue of the convention that $\zeta_\nu = 0$ for $\nu < 0$, the sum is unchanged if we extend the range of summation to include all integers $h_1 > h_2 > \cdots > h_k$. We are now in a situation in which Lemma 1 of [1] can be applied. In the present case that lemma asserts that, for even k, $B_{0k}(x)$ is the Pfaffian of the skew-symmetric $k \times k$ matrix $D_k = (d_{j-i})$, where $d_\nu = c_0 + 2(c_1 + c_2 + \cdots + c_{\nu-1}) + c_\nu$, and $c_\nu = \sum_{n=0}^\infty \zeta_n\zeta_{n+\nu}$. For odd k, $B_{0k}(x)$ is the Pfaffian of the $(k+1)\times(k+1)$ matrix

$$D_k' = \begin{pmatrix} 0 & s \\ -s & D_{k-1}\end{pmatrix}$$

obtained by bordering D_{k-1} with a zero, a row of s's and a column of $-s$'s, where $s = \sum_{n=0}^{\infty} \zeta_n$.

Our next task is to evaluate the quantities s and d_ν. We have

$$D_1' = \begin{pmatrix} 0 & s \\ -s & 0 \end{pmatrix},$$

and, since $B_{01}(x)$ is the Pfaffian of D_1', it follows that $s = B_{01}(x)$. On the other hand, it is clear from the definition that $B_{01}(x) = Q_0(x)$, and so we have proved that $s = Q_0(x)$. Now

$$c_\nu = \sum_{n=0}^{\infty} \frac{x^{n^2+(n+\nu)^2}}{(1-x^2)(1-x^4)\cdots(1-x^{2n})(1-x^2)(1-x^4)\cdots(1-x^{2n+2\nu})}.$$

Putting $y = x^2$, we get

$$c_\nu = x^{\nu^2} \sum_{n=0}^{\infty} \frac{y^{n(n+\nu)}}{(1-y)\cdots(1-y^n)(1-y)\cdots(1-y^{n+\nu})}.$$

It was shown on p. 91 of [1] that the sum of the series on the right is $P(y)$, and therefore $c_\nu = x^{\nu^2} P(x^2)$. It follows that

$$d_\nu = [1 + 2(x + x^4 + \cdots + x^{(\nu-1)^2}) + x^{\nu^2}]\, P(x^2).$$

What now remains is to evaluate the Pfaffians of the matrices D_k and D_k'. We will do this separately for even and odd k in the next two sections.

4. Evaluation of $B_{0k}(x)$ for Even k

If $k = 2m$, we know that $B_{0k}(x)$ is the Pfaffian of D_k, and by Lemma 1 this is equal to the $m \times m$ determinant $\det(t_{i,j})$, where

$$t_{i,j} = d_{|i-j|+1} + d_{|i-j|+3} + \cdots + d_{i+j-1}.$$

Subtract the i-th row of T from the $(i+1)$-st row ($i = 1,\ldots, m-1$), and subtract the j-th column of the resulting matrix T^* from the $(j+1)$-st column ($j = 1,\ldots, m-1$). This gives the matrix

$$\begin{aligned} T^{**} &= P(x^2) \begin{bmatrix} 1+x & x+x^4 & \cdots\ x^{(m-1)^2}+x^{m^2} \\ x+x^4 & 1+x^9 & \cdots\ x^4+x^{(m+1)^2} \\ \vdots & & \\ x^{(m-1)^2}+x^{m^2} & x^{(m-2)^2}+x^{(m+1)^2} & \cdots\ 1+x^{(2m-1)^2} \end{bmatrix} \\ &= P(x^2)U, \end{aligned}$$

say. Clearly, $\det T = \det T^{**} = P(x^2)^m \det U$. From the i-th row of the determinant $\det U$ we remove the factor $(1 + x^{2i-1})$ $(i = 1,\dots, m)$. This yields

$$\det U = \det V \prod_{i=1}^{m} (1 + x^{2i-1}),$$

where V is the matrix $(v_{i,j})$ with

$$v_{i,j} = \frac{x^{(i-j)^2} + x^{(i+j-1)^2}}{1 + x^{2i-1}}.$$

Next multiply the j-th column of V by x^{2j} and add the result to the $(j + 1)$-st column. An easy calculation shows that the resulting matrix is $W = (w_{i,j})$ where

$$w_{i,1} = x^{(i-1)^2}, \qquad \text{and} \qquad w_{i,j} = x^{(i-j)^2} + x^{(k+j-1)^2-2i+1}$$

for $j > 1$. We now multiply the i-th row of W by $x^{2i(i-1)}$ and the j-th column by $x^{-2j(j-1)}$ $(i, j = 1,\dots, m)$. This operation does not change the determinant; it transforms W into the matrix

$$Y = \begin{bmatrix} 1 & x^{-1} + x & x^{-2} + x^2 \cdots & x^{-m+1} + x^{m-1} \\ x^3 & 1 + x^6 & x^{-3} + x^9 \cdots & x^{-3m+6} + x^{3m} \\ x^{10} & x^5 + x^{15} & 1 + x^{20} \cdots & x^{-5m+15} + x^{5m+5} \\ \vdots & & & \\ x^{(m-1)(2m-1)} & \cdots & \cdots & 1 + x^{(2m-2)(2m-1)} \end{bmatrix}.$$

The corollary of Lemma 2 is now applicable, since

$$Y = E(x, x^3, x^5,\dots, x^{2m-1}).$$

Thus we have

$$\det Y = \prod_{1 \leqslant i \leqslant j \leqslant m} (1 - x^{-2i+2j})(1 - x^{2i+2j+2}).$$

Combining all the above transformations, we find that, for $k = 2m$,

$$B_{0k}(x) = P(x^2)^m \prod_{i=1}^{m} (1 + x^{2i-1}) \prod_{1 \leqslant i \leqslant j \leqslant m} (1 - x^{-2i+2j})(1 - x^{2i+2j-2}).$$

Let $f_m(\nu)$ be the total number of representations of ν in either of the forms $\nu = -i + j$ or $\nu = i + j - 1$, where $1 \leqslant i < j \leqslant m$. Clearly $f_1(\nu) = 0$. Moreover, $f_{m+1}(\nu) = f_m(\nu)$ unless ν is of the form $-i + (m + 1)$ or

$i + (m + 1) - 1$, where $1 \leqslant i \leqslant m$. From this we see that $f_{m+1}(\nu) = f_m(\nu)$ for $\nu > 2m$, while $f_{m+1}(\nu) = f_m(\nu) + 1$ for $1 \leqslant \nu \leqslant 2m$. The solution of this recurrence is

$$f_m(\nu) = \left[\frac{2m - \nu}{2}\right].$$

We may therefore simplify (9) to obtain

$$B_{0k}(x) = P(x^2)^m \prod_{j=1}^{m} (1 + x^{2j-1}) \prod_{\nu=1}^{2m-2} (1 - x^{2\nu})^{[(2m-\nu)/2]},$$

which is clearly equivalent to (7).

5. Evaluation of $B_{0k}(x)$ for Odd k

If $k = 2m + 1$, then $B_{0k}(x)$ is the Pfaffian of the $(k + 1) \times (k + 1)$ matrix

$$D_k' = \begin{pmatrix} 0 & s \\ -s & D_k \end{pmatrix}.$$

We subtract each row of D_k' from the succeeding row, and then subtract each column of the resulting matrix from the succeeding column. The result is the matrix

$$D_k'' = \begin{bmatrix} 0 & s & 0 & 0 & 0 & \cdots \\ -s & 0 & d_1 & d_2 - d_1 & d_3 - d_2 & \cdots \\ 0 & -d_1 & 0 & 2d_1 - d_2 & -d_1 + 2d_2 + d_3 & \cdots \\ \vdots & & & & 0 & \\ \vdots & & & & & \ddots \\ \vdots & & & & & \end{bmatrix}.$$

If $i, j \geqslant 2$, the entry in the i-th row and y-th column of D_k'' is $-d_{j-i-1} + 2d_{j-i} - d_{j-i+1}$. This can be written in the form e_{j-i}, where $e_\nu = -d_{\nu-1} + 2d_\nu - d_{\nu+1}$. Expanding the determinant of D_k'' by minors of the first row and then by minors of the first column, we find that the Pfaffian of D_k'' is s times the Pfaffian of the $(k - 1) \times (k - 1)$ matrix

$$\begin{bmatrix} 0 & e_1 & e_2 \cdots e_{k-2} \\ -e_1 & 0 & e_1 \cdots e_{k-1} \\ \vdots & & \\ -e_{k-2} & \cdots\cdots\cdots\cdots & 0 \end{bmatrix}.$$

By Lemma 1, this latter Pfaffian is equal to the determinant of the $m \times m$ symmetric matrix $T = (t_{i,j})$, where

$$t_{i,j} = e_{|i-j|+1} + e_{|i-j|+3} + \cdots + e_{i+j-1}\,.$$

Since

$$d_\nu = [1 + 2(x + x^4 + \cdots + x^{(\nu-1)^2}) + x^{\nu^2}]\, P(x^2),$$

we have

$$e_\nu = [x^{(\nu-1)^2} - x^{(\nu+1)^2}]\, P(x^2),$$

from which it follows easily that

$$t_{i,j} = [x^{(i-j)^2} - x^{(i+j)^2}]\, P(x^2).$$

Thus

$$\begin{aligned} T &= P(x^2)\begin{bmatrix} 1 - x^4 & x - x^9 \cdots & x^{(m-1)^2} - x^{(m+1)^2} \\ x - x^9 & 1 - x^{16} \cdots & x^{m^2} - x^{(m+2)^2} \\ \vdots & & \vdots \\ x^{(m-1)^2} - x^{(m+1)^2} & \cdots\cdots\cdots\cdots & 1 - x^{(2m)^2} \end{bmatrix} \\ &= P(x^2)U, \end{aligned}$$

say. Clearly then $\det T = P(x^2)^m \det U$. From the i-th row of the determinant $\det U$ we remove the factor $(1 - x^{4i})$ $(i = 1,\ldots, m)$. This yields

$$\det U = \prod_{i=1}^{m} (1 - x^{4i}) \det V,$$

where V is the matrix $(v_{i,j})$ with

$$v_{i,j} = \frac{x^{(i-j)^2} - x^{(i+j)^2}}{1 - x^{4i}}\,.$$

Next multiply the j-th column of V $(j = 1,\ldots, m-2)$ by x^{4j+4} and subtract from the $(j+2)$-nd column. One easily sees that the resulting matrix is $W = (w_{i,j})$, where $w_{i,1} = x^{(i-1)^2}$, while

$$w_{i,j} = x^{(i-j)^2} + x^{(i+j-1)^2+2j-3}.$$

We now multiply the i-th row of W by x^{i^2-1} and the j-th column by x^{-j^2+1} $(i, j = 1,\ldots, m)$. The resulting matrix $Y = (y_{i,j})$ has the same determinant as W.

We have

$$Y = \begin{bmatrix} 1 & x^{-2} + x^2 & x^{-4} + x^4 & \cdots & x^{-2(m-1)} + x^{2(m-1)} \\ x^4 & 1 + x^8 & x^{-4} + x^{12} & \cdots & x^{-4(m-2)} + x^{4(m-2)} \\ \vdots & & & & \\ x^{2m(m-1)} & \cdots\cdots & & & 1 + x^{4m(m-1)} \end{bmatrix}.$$

The corollary to Lemma 2 is now applicable, since $Y = E(x^2, x^4, \ldots, x^{2m})$. Thus

$$\det Y = \prod_{1 \leqslant i < j \leqslant m} (1 - x^{-2i+2j})(1 - x^{2i+2j}).$$

Putting all the above transformations together, we find that

$$B_{0k}(x) = Q_0(x)\, P(x^2)^m \prod_{i=1}^{m} (1 - x^{4i})$$

$$\cdot \prod_{1 \leqslant i < j \leqslant m} (1 - x^{-2i+2j})(1 - x^{2i+2j}). \tag{10}$$

This can be written in the form

$$B_{0k}(x) = Q_0(x)\, P(x^2)^m \prod_{\nu=1}^{2m} (1 - x^{2\nu})^{g_m(\nu)},$$

where $g_m(\nu)$ is the total number of representations of ν in the form $\nu = -i + j$ or $\nu = i + j$, where $1 \leqslant i \leqslant j \leqslant m$. We have $g_{m+1}(\nu) = g_m(\nu)$ unless $\nu = -i + (m+1)$ or $\nu = i + (m+1)$, where $1 \leqslant i \leqslant m+1$. Thus $g_{m+1}(\nu) = g_m(\nu)$ if $\nu = m+1$ or $\nu > 2m+2$, while $g_{m+1}(\nu) = g_m(\nu) + 1$ if $1 \leqslant \nu \leqslant m$ or $m+2 \leqslant \nu \leqslant 2m+2$. Moreover $g_1(\nu) = 1$ if $\nu = 2$ and 0 otherwise. The solution of this recurrence is

$$g_m(\nu) = \left[\frac{2m - \nu}{2}\right]$$

for $1 \leqslant \nu \leqslant m$ and

$$g_m(\nu) = \left[\frac{2m + 2 - \nu}{2}\right]$$

for $m + 1 \leqslant \nu \leqslant 2m$. Hence (10) can be simplified to give

$$B_{0k}(x) = Q_0(x)\, P(x^2)^m \frac{\prod_{\nu=1}^{2m} (1 - x^{2\nu})^{[(2m+2-\nu)/2]}}{\prod_{\nu=1}^{m} (1 - x^{2\nu})},$$

which is equivalent to (8).

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